On the Convergence of the Hegselmann-Krause System

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Abstract

We study convergence of the following discrete-time non-linear dynamical system: n agents are located in $\mathbb{R}^d$ and at every time step, each moves synchronously to the average location of all agents within a unit distance of it. This popularly studied system was introduced by Krause to model the dynamics of opinion formation and is often referred to as the *Hegselmann-Krause model*. We prove the first polynomial time bound for the convergence of this system in arbitrary dimensions. This improves on the bound of $n^{O(n)}$ resulting from a more general theorem of Chazelle [4]. Also, we show a quadratic lower bound and improve the upper bound for one-dimensional systems to $O(n^3)$.

1 Introduction

Lately, there has been a surge of attention given to network-based dynamical systems, in which agents interact according to local rules via a dynamic communication graph [7]. Much of the previous work has focused on the exogenous case, where the communication topology is decoupled from the evolution of the system. We refer interested readers to [3] for a good overview of research in this area. In the more common, endogenous version, the communication graph changes over time according to the current states. The feedback loop between dynamics and topology creates considerable difficulties, and efforts have been underway to build an algorithmic calculus within the broad framework of *influence systems* [5]. This work investigates the complexity of *Hegselmann-Krause systems* (abbreviated as HK system from now on), a popular model of opinion dynamics that has proven highly influential over the years [1, 8, 9, 11, 13] and stands as the archetype of a diffusive influence system [6]. In the d-dimensional version of the model, each agent has an opinion represented as a point in $\mathbb{R}^d$. Two agents are neighbors if they are within unit distance from each other. At every time step, each agent moves synchronously to the mass center of its neighbors.

HK systems are known to converge [10, 12, 16], meaning that they eventually come to a full stop. The convergence time has been bounded by $n^{O(n)}$ time and conjectured to be polynomial [4].

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It was shown to be $O(n^5)$ in the case $d = 1$ [14]. The contribution of this work is threefold: first, we prove that the convergence time is indeed polynomial in n, regardless of the dimension; second, we lower the one-dimensional bound from $O(n^5)$ to $O(n^3)$; third, we establish a quadratic lower bound, which improves the known bound of $\Omega(n)$ [14]. We also consider noisy variants of the model.

The bidirectionality of the system plays a crucial role in the proof, as it should. Indeed, it is known from [5] that allowing different radii and averaging weights for the agents can prevent the convergence of the communication graph. Our proofs are an elementary mix of geometric and algebraic techniques. Much of the current technology for HK systems centers around products of stochastic matrices. This work injects a geometric perspective that, we believe, will be necessary for further progress on the more difficult directional case.

2 Preliminaries

We formally define the discrete-time *HK system in dimension* d as follows. There are n agents. For every $t \in \mathbb{Z}^{\geqslant 0}$ and for every $i \in [n]$, the *position* of agent i at time t is $\mathbf{x}_i(t) \in \mathbb{R}^d$. The positions at $t = 0$ are given and, thereafter, are updated synchronously according to the following rule:

$$\mathbf{x}_i(t+1) = \frac{\sum\limits_{j:\|\mathbf{x}_i(t)-\mathbf{x}_j(t)\|\leqslant 1} \mathbf{x}_j(t)}{\sum\limits_{j:\|\mathbf{x}_i(t)-\mathbf{x}_j(t)\|\leqslant 1} 1} \tag{1}$$

Here, $\|\cdot\|$ denotes the Euclidean norm. We say that agents i and j are *neighbors* at time t if $\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| \leqslant 1$, and we denote the *neighborhood* at time t of agent i by $\mathcal{N}_i(t) = \{j : i \text{ and } j \text{ neighbors at time } t\}$. Note that if $i \in \mathcal{N}_j(t)$, then $j \in \mathcal{N}_i(t)$. Also for a given $\mathbf{x} \in \mathbb{R}^d$, we define the *weight of* $\mathbf{x}$ *at time* t to be $w_t(\mathbf{x}) = |\{i : \mathbf{x}_i(t) = \mathbf{x}\}|$. Note that at any given t, $0 \leqslant w_t(\mathbf{x}) \leqslant n$ for all $\mathbf{x} \in \mathbb{R}^d$ and that the sum of all weights equals n. Also, let the *weight of agent* i *at time* t denote $w_t(\mathbf{x}_i(t))$. The weight of an agent is monotonically non-decreasing with time. The system is said to have *converged* at time t if $\mathbf{x}_i(t+1) = \mathbf{x}_i(t)$ for all $i \in [n]$.

In the case of one dimension, one can observe that order is preserved:

Proposition 2.1 (Lemma 2 in [11]). *The HK system in dimension 1 preserves order of positions. That is, for any $i, j \in [n]$, if $x_i(0) \leqslant x_j(0)$, then $x_i(t) \leqslant x_j(t)$ for all $t > 0$.* □

For this reason, in one dimension, we can number the agents from 1 to n such that $x_1(t) \leqslant x_2(t) \leqslant \cdots \leqslant x_n(t)$ for all $t \geqslant 0$. The one-dimensional system also has the following decomposability property:

Proposition 2.2 (Proposition 2 in [3]). *Suppose $d = 1$. If at time t, and for some $i \in [n]$, $|x_{i+1}(t) - x_i(t)| > 1$, then for all $t' > t$ also, $|x_{i+1}(t') - x_i(t')| > 1$. So, the system can be decomposed into two subsystems, one consisting of agents $\{1, \ldots, i\}$ and the other of agents $\{i+1, \ldots, n\}$, each evolving independently after time t.* □

Therefore, if at time t, an agent has no neighbor strictly to its left and no neighbor strictly to its right, it never moves subsequently, and we say that the agent has *frozen* at time t.

This decomposability property is not true in higher dimensions. For instance, consider the two-dimensional HK system with agents at positions $(0, 0)$, $(1, 0)$ and $(1/2, 1)$; the third agent does not neighbor any other agent at $t = 0$, but this is not so at $t = 1$.

3 Convergence in One Dimension

Theorem 3.1. *The HK system in* 1 *dimension converges within* $O(n^3)$ *time steps.*

Proof. Recall that we number the agents such that $x_i(t) \leqslant x_{i+1}(t)$ for all t and all $i \in [1, n-1]$. Suppose the system has not already converged at time t, and let $\ell(t)$ denote the leftmost non-frozen agent, i.e., the least $\ell \geqslant 1$ such that $\{x_i(t) : i \in \mathcal{N}_\ell(t)\} \neq \{x_\ell(t)\}$. Note that at time t, agent $\ell(t)$ must have at least one neighbor strictly to its right and no neighbor strictly to its left (because any neighbor strictly to the left would violate minimality of ℓ).

The following lemma is the heart of our proof.

Lemma 3.2. *For every* $t \geqslant 0$*, by time* $t+2$*, agent* $\ell(t)$ *increases in weight, or gets frozen, or moves to the right by at least* $\frac{1}{2n^2}$*.*

Proof. Fix t, let $\ell = \ell(t)$, and let $r = \min\{j : x_j(t) > x_\ell(t)\}$ be the leftmost agent that is strictly right of agent ℓ. Agent r is a neighbor of ℓ because ℓ has at least one neighbor strictly to its right. If $\mathcal{N}_\ell(t) = \mathcal{N}_r(t)$, agent ℓ at time $t+1$ moves to the same location as agent r does and increases its weight. So, suppose otherwise. Since ℓ has no neighbor strictly to its left, r also does not have any neighbor strictly left of ℓ. Hence, there must exist $s \in \mathcal{N}_r(t) \setminus \mathcal{N}_\ell(t)$ strictly to the right of r.

We now show $x_r(t+1) \geqslant x_\ell(t) + \frac{1}{n}$. Observe that $x_s(t) - x_\ell(t) > 1$. So, if $\delta = x_r(t) - x_\ell(t)$ and $k = |\mathcal{N}_r(t)|$, then agent r moves to the left by at most:

$$\frac{\delta \cdot (k-1) - (1-\delta) \cdot 1}{k} = \delta - \frac{1}{k} \leqslant \delta - \frac{1}{n}$$

If $x_\ell(t+1) \geqslant x_\ell(t) + \frac{1}{2n}$, we are already done. Otherwise, $x_r(t+1) - x_\ell(t+1) \geqslant \frac{1}{2n}$. By Proposition 2.2, agent ℓ still has no neighbor strictly to its left at time $t{+}1$. If $x_r(t{+}1) - x_\ell(t{+}1) > 1$, then agent ℓ gets frozen at time $t+1$. Otherwise, $\frac{1}{2n} \leqslant x_r(t+1) - x_\ell(t+1) \leqslant 1$, and so:

$$x_\ell(t+2) - x_\ell(t+1) \geqslant \frac{1}{2n|\mathcal{N}_\ell(t+1)|} \geqslant \frac{1}{2n^2}$$

□

We can assume $x_n(0) - x_1(0) \leqslant n$ without loss of generality, because otherwise, by Proposition 2.2, the system can be decomposed into independently evolving subsystems. So, $x_n(t) - x_1(t) \leqslant n$ for all t. Now, apply Lemma 3.2 at $t = 0, 2, 4, \ldots$ as long as the system has not converged and $\ell(t)$ exists. $\ell(t)$ can increase in weight only at most n times. Also, because $\ell(t)$ is non-decreasing with t, $x_{\ell(t+2)}(t+2) \geqslant x_{\ell(t)}(t+2)$ and, hence, the third case in Lemma 3.2 can occur only at most $2n^3$ times. Thus, after $t > 2(n + 2n^3)$, Lemma 3.2 cannot be applied, and the system must have converged. □

3.1 Noisy neighborhoods

In this subsection, we study an extension to a noisy version of the HK model. We consider the following system HK_η where $\eta \in (0,1)$ is a parameter. There are n agents, and each agent has an

associated *left-neighborhood parameter* η_i that is in the interval $(0, \eta)$. The positions at $t = 0$ are given, and then, the positions are updated according to the following rule:

$$x_i(t+1) = \frac{\sum\limits_{j: -1+\eta_i \leqslant x_j(t) - x_i(t) \leqslant 1} x_j(t)}{\sum\limits_{j: -1+\eta_i \leqslant x_j(t) - x_i(t) \leqslant 1} 1} \tag{2}$$

One can interpret the η_i's as noise acting on the left side of each agent's neighborhood. We can prove the following extension of Theorem 3.1 to HK_η.

Theorem 3.3. *For fixed $\eta \in (0,1)$, the HK_η system converges within $O(n^3)$ time steps.*

Proof. The proof proceeds similarly to that of Theorem 3.1. We say that agent j is a neighbor of agent i at time t if $-1 + \eta_i \leqslant x_j(t) - x_i(t) \leqslant 1$. Conceptually, the main complicating issue is that being neighbors is no longer a symmetric relation. Also, Proposition 2.1 is no longer true as order may not be preserved by the dynamics.

Let $\ell(t)$ denote the leftmost agent at time t that neighbors at least one agent positioned strictly to its right. Notice that this implies there is no k such that $0 < x_{\ell(t)} - x_k \leqslant 1$, because agent k would neighbor $\ell(t)$ and hence violate the minimality of $\ell(t)$. We prove the following analog of Lemma 3.2.

Lemma 3.4. *For every $t \geqslant 0$, by time $t+2$, agent $\ell(t)$ increases in weight, or gets frozen, or moves to the right by at least $\frac{1}{2n^2}$.*

Proof. Let $\ell = \ell(t)$ and let r be the leftmost neighbor of ℓ. If $x_r(t) - x_\ell(t) > 1 - \eta_r$, then $x_\ell(t+1) > {}^{(1-\eta_r)}/_{n} \geqslant {}^{(1-\eta)}/_{n}$, and we are already done. So, assume otherwise.

Then, agent ℓ is a neighbor of agent r. Since ℓ has no neighbor strictly to its left within a distance of 1, r also has no neighbor strictly to the left of ℓ. Now, we can use the same argument as in the proof of Lemma 3.2 to argue that either agent ℓ freezes at time $t+1$ or at time $t+2$, it moves to the right by at least $\frac{1}{2n^2}$. $\square$

The rest of the proof can be finished in exactly the same way as the proof of Theorem 3.1. $\square$

By symmetry, Theorem 3.3 also applies when the right side of each agent's neighborhood is perturbed and the left side is fixed. The case when both sides are perturbed remains open; convergence for such heterogeneous HK systems is conjectured [15].

4 Convergence in Higher Dimensions

In this section, we consider the HK system in d dimensions with n agents and $d \geqslant 2$. We show that the convergence time is polynomial in both n and d.

The proof follows from a sequence of lemmas. The first lemma asserts that there is some vector $\mathbf{a}$ such that taking the projection of agents on $\mathbf{a}$ does not bring two non-neighbors too close together.

Lemma 4.1. *For every t, there exists a unit vector $\mathbf{a}$ such that for any two agents i, j, we have $\left|\mathbf{a} \cdot \frac{\mathbf{x}_i(t) - \mathbf{x}_j(t)}{\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\|}\right| = \Omega(n^{-2} d^{-1})$.*

Proof. Let $\mathbf{d}_{i,j} = \frac{\mathbf{x}_i(t)-\mathbf{x}_j(t)}{\|\mathbf{x}_i(t)-\mathbf{x}_j(t)\|}$. We can view $\mathbf{d}_{i,j}$ as a point on S^{d-1}, the unit ball in $\mathbb{R}^d$. The set of points on S^{d-1} with dot product with $\mathbf{d}_{i,j}$ of absolute value at most $O(n^{-2}d^{-1})$ has area $\frac{\pi^{(d-2)/2}}{n^2 d \cdot \Gamma((d-2)/2)}$. The area of S^{d-1} is $\frac{2\pi^{d/2}}{\Gamma(d/2)}$. There are $\binom{n}{2}$ pairs i, j, and so, by a volume argument, there exists a point $\mathbf{a} \in S^{d-1}$ such that for all i and j, $|\mathbf{d}_{i,j} \cdot \mathbf{a}| = \Omega(n^{-2}d^{-1})$. □

The next lemma analyzes the one-dimensional system formed by projecting onto $\mathbf{a}$ and shows a lower bound on the total movement in each step.

Lemma 4.2. *Assume that at time t, the system has not converged. One of the following cases happens.*

- *Two agents move to the same location at time $t+1$.*
- *Some agent moves by at least $\Omega(n^{-4}d^{-1})$*

Proof. Let $\mathbf{a}$ be a unit vector satisfying Lemma 4.1 and j be the agent with minimum $\mathbf{x}_j(t) \cdot \mathbf{a}$ among agents with neighbor at time t. Consider two cases.

Case 1. j has a neighbor i such that $(\mathbf{x}_i(t) - \mathbf{x}_j(t)) \cdot \mathbf{a} = \Omega(n^{-3}d^{-1})$. Because $(\mathbf{x}_k(t) - \mathbf{x}_j(t)) \cdot \mathbf{a} \geqslant 0\ \forall k$, j must move by $\Omega(n^{-4}d^{-1})$.

Case 2. All neighbors i of j satisfy $(\mathbf{x}_i(t) - \mathbf{x}_j(t)) \cdot \mathbf{a} = O(n^{-3}d^{-1})$. If none of them has any neighbor that is not j or j's neighbors, then they all move to the same location at time $t+1$. Otherwise, some neighbor i of j has a neighbor k that is not a neighbor of j. Then $(\mathbf{x}_k(t) - \mathbf{x}_j(t)) \cdot \mathbf{a} = \Omega(n^{-2}d^{-1})$. We have

$$\mathbf{x}_i(t+1) \cdot \mathbf{a} \geqslant \mathbf{x}_i(t) \cdot \mathbf{a} + \frac{(\mathbf{x}_k(t) - \mathbf{x}_j(t)) \cdot \mathbf{a} + (n-1)(\mathbf{x}_j(t) - \mathbf{x}_i(t)) \cdot \mathbf{a}}{n} \geqslant \mathbf{x}_i(t) + \Omega(n^{-3}d^{-1})$$

Thus i moves by at least $\Omega(n^{-3}d^{-1})$. □

We now use the following special case of a result from [17] to show the existence of a potential function which is strictly decreasing as long as some agent moves. The proof is included for completeness.

Theorem 4.3 ([17], Theorem 2)**.** *Let $f_{i,j}^t : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be defined as follows: $f_{i,j}^t(\mathbf{z}, \mathbf{z}') = \|\mathbf{z} - \mathbf{z}'\|^2$ if $\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| < 1$ and $f_{i,j}^t(\mathbf{z}, \mathbf{z}') = 1$ otherwise. Let $V(t) = \sum_{i,j \in [n]} f_{i,j}^t(\mathbf{x}_i(t), \mathbf{x}_j(t))$. Then V is non-increasing along the trajectory of the system. In fact:*

$$\begin{aligned} V(t) - V(t+1) &\geqslant \sum_{i,j : f_{i,j}^t \not\equiv 1} f_{i,j}^t(\mathbf{x}_i(t+1) - \mathbf{x}_i(t), \mathbf{x}_j(t) - \mathbf{x}_j(t+1)) \\ &\geqslant \sum_i 4\|\mathbf{x}_i(t+1) - \mathbf{x}_i(t)\|^2 \end{aligned}$$

Proof. First, notice that $f_{i,j}^{t+1}(\mathbf{x}_i(t+1), \mathbf{x}_j(t+1)) \leqslant f_{i,j}^t(\mathbf{x}_i(t+1), \mathbf{x}_j(t+1))\ \forall i, j$. Next, we will show

$$\sum_{i,j} f_{i,j}^t(\mathbf{x}_i(t), \mathbf{x}_j(t)) - f_{i,j}^t(\mathbf{x}_i(t+1), \mathbf{x}_j(t+1)) = \sum_{i,j\in[n]:f_{i,j}^t \not\equiv 1} f_{i,j}^t(\mathbf{x}_i(t+1) - \mathbf{x}_i(t), \mathbf{x}_j(t) - \mathbf{x}_j(t+1))$$

Let $F : (\mathbb{R}^d)^n \times (\mathbb{R}^d)^n \to \mathbb{R}$ be defined as $F(\mathbf{x}, \mathbf{y}) = \sum_{i,j:f_{i,j}\not\equiv 1} f_{i,j}^t(\mathbf{x}_i, \mathbf{y}_j)$. Then $\mathbf{x}(t+1) = \operatorname{argmin}_{\mathbf{y}} F(\mathbf{x}(t), \mathbf{y})$. We can also view F as 2 matrices A, B and $F(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T A\mathbf{x} + \mathbf{y}^T A\mathbf{y} - 2\mathbf{x}^T B\mathbf{y}$ where A, B are symmetric and A is positive semidefinite. We have $\mathbf{x}(t+1) = A^{-1}B\mathbf{x}(t)$.

$$\begin{aligned} F(\mathbf{x}, \mathbf{x}) - F(A^{-1}B\mathbf{x}, A^{-1}B\mathbf{x}) &= 2\mathbf{x}^T(BA^{-1}(A-B)A^{-1}B + A - B)\mathbf{x} \\ &= 2\mathbf{x}^T(BA^{-1}B - BA^{-1}BA^{-1}B + A - B)\mathbf{x} \\ &= 2\mathbf{x}^T(I - BA^{-1})(A+B)(I - A^{-1}B)\mathbf{x} \\ &= F((I - A^{-1}B)\mathbf{x}, -(I - A^{-1}B)\mathbf{x}) \end{aligned}$$

□

Theorem 4.4. *The system converges in* poly(n, d) *time.*

Proof. There are at most n time steps where two agents move to the same place. In all other time steps, by Lemma 4.2, some agent moves by at least $\Omega(n^{-4}d^{-1})$. We have $V(0) \leqslant n^2$ and $V(t) \geqslant 0\ \forall t$. Therefore, by Theorem 4.3, the number of time steps before the system converges is $O(n^{10}d^2)$. □

5 Lower bound

In this section, we give an instance of an HK system that requires $\Omega(n^2)$ steps to converge. Our example is in two dimensions, and we know of no example that takes longer time even when the number of dimensions is large. The previous best lower bound on the convergence time [14] was $\Omega(n)$, which is achieved by n points on a line with unit distance spacing between each pair of consecutive points. This is still the worst example we know of in one dimension.

Our lower bound is achieved by the following system. There are n agents in the system and initially, they are located at vertices of a regular n-gon with side length $l_1 = 1$. Label the agents clockwise around the polygon from 0 to $n-1$. Let O be the center of the polygon and A_i be the location of agent i. For notational convenience, all index computation is done modulo n (so if $i = 0$ then A_{i-1} is A_{n-1}).

Theorem 5.1. *The system described above requires at least* $\Omega(n^2)$ *steps to converge.*

Proof. We will prove by induction that for all $t \leqslant n^2/28$, A_i's are vertices of a regular n-gon with side length $l_t \geqslant (1 - \frac{14}{n^2})^t$.

By the initial state of the system, the invariant holds for $t = 1$. Assume that it holds before step $t = k$, we will show it holds before step $t = k + 1 \leqslant n^2/28$.

We analyze the movement of agent i from A_i to A_i' in one step (see Fig. 1). Note that $\angle A_iOA_{i+1} = \frac{2\pi}{n}$. For any $j \notin \{i-1, i, i+1\}$, we have

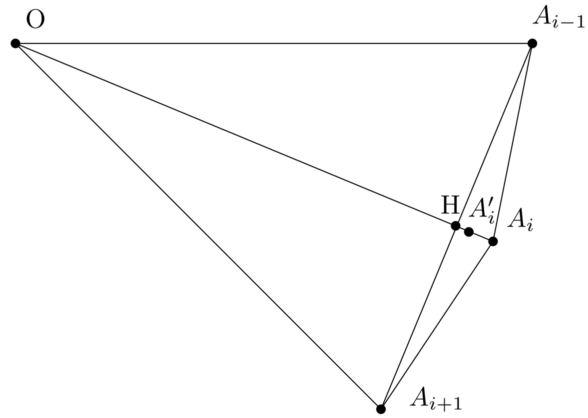

Figure 1: In one time step, agent i moves from A_i to A_i', the centroid of A_i, A_{i-1}, A_{i+1}

$$
\begin{aligned}
A_iA_j &= \frac{OA_i \sin(\angle OA_iA_j)}{\sin(\angle A_iOA_j)} \\
&\geqslant \frac{OA_i \sin(\frac{4\pi}{n})}{\sin(\frac{\pi(n-4)}{2n})} \\
&= \frac{l_t \sin(\frac{\pi(n-2)}{2n}) \sin(\frac{4\pi}{n})}{\sin(\frac{2\pi}{n}) \sin(\frac{\pi(n-4)}{2n})} \\
&\geqslant 2l_k \left(1 - \frac{\pi^2}{2n^2}\right)\left(1 - \frac{4\pi^2}{n^2}\right) \\
&> 1
\end{aligned}
$$

Thus, for $j \notin \{i-1, i, i+1\}$, agents i and j are not neighbors. A_i moves to the centroid of A_i, A_{i-1}, A_{i+1} and by symmetry, the locations of the agents are vertices of a new regular n-gon centered at O and with a smaller side length l_{k+1}. We have

$$
\begin{aligned}
l_{k+1} &= \frac{OA_i'}{OA_i} l_k \\
&= \left(1 - \frac{2HA_i}{3OA_i}\right) l_k \\
&= \left(1 - \frac{2l_k \sin(\frac{\pi}{n}) \sin(\frac{2\pi}{n})}{3l_k \sin(\frac{\pi(n-2)}{2n})}\right) l_k \\
&\geqslant \left(1 - \frac{14}{n^2}\right) l_k
\end{aligned}
$$

Thus, the system requires at least $n^2/28$ steps to converge. $\square$

6 Discussion

In this paper, we analyzed the convergence rate of the homogeneous HK system in arbitrary dimensions. The system is shown to converge in polynomial time, but can take at least a quadratic number of steps in the worst case. Getting tight bounds on the convergence time of the system, even in just one dimension, remains an interesting open problem.

A particularly interesting challenge is to analyze the heterogeneous version of the HK system, i.e. when the neighborhood radii of all the agents are not necessarily the same [15]. New ideas are needed to understand the behavior of this system in particular, and directional systems in general. Our analysis of a noisy variant of homogeneous HK system is a step towards studying more complicated directional systems.

Beyond convergence rate, the behavior of the homogeneous HK system is still full of mysteries. Most notable perhaps is the 2R conjecture [2]: when agents are drawn uniformly at random on an interval, they converge to clusters at distance close to twice the minimum possible inter-cluster distance. Resolving this conjecture remains well beyond our understanding of the system.

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